

# Horoball packings to the totally asymptotic regular simplex in the hyperbolic $n$ -space \*

Jenő Szirmai

Budapest University of Technology and Economics

Institute of Mathematics, Department of Geometry

H-1521 Budapest, Hungary

Email: szirmai@math.bme.hu

(December 12, 2011)

## Abstract

In [23] we have generalized the notion of the simplicial density function for horoballs in the extended hyperbolic space  $\overline{\mathbf{H}}^n$ , ( $n \geq 2$ ), where we have allowed *congruent horoballs in different types* centered at the various vertices of a totally asymptotic tetrahedron. By this new aspect, in this paper we study the locally densest horoball packing arrangements and their densities with respect to totally asymptotic regular tetrahedra in hyperbolic  $n$ -space  $\overline{\mathbf{H}}^n$  extended with its absolute figure, where the ideal centers of horoballs give rise to vertices of a totally asymptotic regular tetrahedron.

We will prove that, in this sense, *the well known Böröczky density upper bound for "congruent horoball" packings of  $\overline{\mathbf{H}}^n$  does not remain valid for  $n \geq 4$* , but these locally optimal ball arrangements do not have extensions to the whole  $n$ -dimensional hyperbolic space. Moreover, we determine an explicit formula for the density of the above locally optimal horoball packings, allowing horoballs in different types.

---

\*Mathematics Subject Classification 2010: 52C17, 52C22, 52B15.

Key words and phrases: Hyperbolic geometry, horoball packings, optimal simplicial density.

# 1 Introduction

We consider the horospheres and their bodies, the horoballs. A horoball packing  $\mathcal{B} = \{B\}$  of  $\overline{\mathbf{H}}^n$  is an arrangement of non-overlapping horoballs  $B$  in  $\overline{\mathbf{H}}^n$ . The notion of local density of the usual ball packing can be extended for horoball packings  $\mathcal{B}$  of  $\overline{\mathbf{H}}^n$ . Let  $B \in \mathcal{B}$ , and  $P \in \overline{\mathbf{H}}^n$  an arbitrary point. Then,  $\rho(P, B)$  is defined to be the length of the unique perpendicular from  $P$  to the horosphere  $S_h$  bounding  $B$ , where again  $\rho(P, B)$  is taken negative for  $P \in B$ . The Dirichlet–Voronoi cell (shortly D-V cell)  $\mathcal{D}(B)$  of  $B$  in  $\mathcal{B}$  is defined to be the convex body

$$\mathcal{D}_h = D(\mathcal{B}, B) := \{P \in \mathbf{H}^n \mid \rho(P, B) \leq \rho(P, B'), \forall B' \in \mathcal{B}\}. \quad (1.1)$$

Since both,  $B$  and  $\mathcal{D}$ , are of infinite volume, the usual concept of local density has to be modified. Let  $Q \in \partial\mathbf{H}^n$  denote the base point (ideal center at the infinity) of  $B$ , and interpret  $S$  as a Euclidean  $(n-1)$ -space. Let  $B_{n-1}(R) \subset S_h$  be an  $n-1$ -ball with center  $C \in S_h$ . Then,  $Q \in \partial\mathbf{H}^n$  and  $B_{n-1}(R)$  determine a convex cone  $C_n(R) := \text{cone}(B_{n-1}^Q(R)) \in \overline{\mathbf{H}}^n$  with apex  $Q$  consisting of all hyperbolic geodesics through  $B_{n-1}(R)$  with limiting point  $Q$ . With these preparations, the local density  $\delta_n(B, \mathcal{B})$  of  $B$  to  $\mathcal{D}$  is defined by

$$\delta_n(\mathcal{B}, B) := \overline{\lim}_{R \rightarrow \infty} \frac{\text{vol}(B \cap C_n(R))}{\text{vol}(\mathcal{D} \cap C_n(R))}, \quad (1.2)$$

and this limes superior is independent of the choice of the center  $C$  of  $B_{n-1}(R)$ .

*In [23] we have refined the notion of the „congruent” horoballs in a horoball packing to the horoballs of the ”same type” because the horoballs are in general congruent in the hyperbolic space  $\overline{\mathbf{H}}^n$ .*

*Two horoballs in a horoball packing are in the ”same type” if and only if the local densities of the horoballs to the corresponding cell (e.g. D-V cell; or ideal simplex, later on) are equal. If we assume that the „horoballs belong to the same type”, then by analytical continuation, the well known simplicial density function on  $\overline{\mathbf{H}}^n$  can be extended from  $n$ -balls of radius  $r$  to the case  $r = \infty$ , too. Namely, in this case consider  $n+1$  horoballs  $B$  which are mutually tangent. The convex hull of their base points at infinity will be a totally asymptotic or ideal regular simplex  $T_{reg}^\infty \in \overline{\mathbf{H}}^n$  of finite volume. Hence, in this case it is legitimate to write*

$$d_n(\infty) = (n+1) \frac{\text{vol}(B) \cap T_{reg}^\infty}{\text{vol}(T_{reg}^\infty)}. \quad (1.3)$$

Then for a horoball packing  $\mathcal{B}$ , there is an analogue of ball packing, namely (cf. [4], Theorem 4)

$$\delta_n(\mathcal{B}, B) \leq d_n(\infty), \forall B \in \mathcal{B}. \quad (1.4)$$

**Remark 1.1** *The upper bound  $d_n(\infty)$  ( $n = 2, 3$ ) is attained for a regular horoball packing, that is, a packing by horoballs which are inscribed in the cells of a regular honeycomb of  $\overline{\mathbf{H}}^n$ . For dimensions  $n = 2$ , there is only one such packing. It belongs to the regular tessellation  $\{\infty, 3\}$ . Its dual  $\{3, \infty\}$  is the regular tessellation by ideal triangles all of whose vertices are surrounded by infinitely many triangles. This packing has in-circle density  $d_2(\infty) = \frac{3}{\pi} \approx 0.95493\dots$*

*In  $\overline{\mathbf{H}}^3$  there is exactly one horoball packing whose Dirichlet–Voronoi cells give rise to a regular honeycomb described by the Schläfli symbol  $\{6, 3, 3\}$ . Its dual  $\{3, 3, 6\}$  consists of ideal regular simplices  $T_{reg}^\infty$  with dihedral angle  $\frac{\pi}{3}$  building up a 6-cycle around each edge of the tessellation.*

**If horoballs of different types at the various ideal vertices are allowed,** then we can generalize the notion of the simplicial density function [23]:

**Definition 1.2** *We consider an arbitrary totally asymptotic simplex  $T = E_0E_1E_2E_3\dots E_n$  in the  $n$ -dimensional hyperbolic space  $\overline{\mathbf{H}}^n$ . Centers of horoballs are required to lie at vertices of  $T$ . We allow horoballs  $(B_i, i = 1, 2, \dots, n)$  of different types at the various vertices and require to form a packing, moreover we assume that*

$$\text{card}(B_i \cap [E_{i_0}E_{i_1}\dots E_{i_{n-1}}]) \leq 1, \quad i_j \neq i, \quad j \in \{0, 1, \dots, n-1\}.$$

*(The hyperplane of points  $E_{i_0}, E_{i_1}, \dots, E_{i_{n-1}}$  is denoted by  $[E_{i_0}E_{i_1}\dots E_{i_{n-1}}]$  may touch the horoball  $B_{i_n}$ .) The generalized simplicial density function for the above simplex and horoballs is defined as*

$$\delta(\mathcal{B}) = \frac{\sum_{i=0}^n \text{vol}(B_i \cap T)}{\text{vol}(T)}.$$

In [23] I have studied the locally densest horoball packing arrangements and their densities with respect to totally asymptotic tetrahedra  $T(\alpha)$  in hyperbolic 3-space  $\overline{\mathbf{H}}^3$ , where the ideal centers of horoballs give rise to vertices of  $T(\alpha)$ . Moreover, I have proved that, in this sense, *the well known Böröczky density upper bound for "congruent horoball" packings of  $\overline{\mathbf{H}}^3$  does not remain valid.*

In [10] we have proved that the known Böröczky–Florian density upper bound for "congruent horoball" packings of  $\overline{\mathbf{H}}^3$  remains valid for the class of fully asymptotic Coxeter tilings, even if packing conditions are relaxed by allowing horoballs of different types under prescribed symmetry groups. The consequences of this remarkable result are discussed for various Coxeter tilings (see [10]), and we have obtained four different optimal horoball packings with the maximal density.

Now, the main problem is to find the locally densest horoball packing related to the  $n$ -dimensional ( $n \geq 4$ ) totally asymptotic *regular* simplex while allowing different types of horoballs ( $B_i$ ) to be centered at the vertices  $E_i$  ( $i = 0, 1, 2, \dots, n$ ) of the simplex, such that the density  $\delta(\mathcal{B})$  (see Definition 1.2) of the corresponding horoball arrangement is maximal. In this case the horoball arrangement  $\mathcal{B}$  is said to be *locally optimal*.

We will prove that, in this sense, the well known Böröczky density upper bound for "congruent horoball" packings of  $\overline{\mathbf{H}}^n$  ( $n \geq 4$ ) does not remain valid, but these locally optimal ball arrangements do not have extensions to the whole  $n$ -dimensional hyperbolic space.

For example, the density of this locally densest packing in  $\overline{\mathbf{H}}^4$  is  $\approx 0.77038$  which is larger than the Böröczky density upper bound  $\approx 0.73046$ .

## 2 Computations in projective model

For  $\overline{\mathbf{H}}^n$   $n \geq 2$  we use the projective model in Lorentz space  $\mathbf{E}^{1,n}$  of signature  $(1, n)$ , i.e.  $\mathbf{E}^{1,n}$  is the real vector space  $\mathbf{V}^{n+1}$  equipped with the bilinear form of signature  $(1, n)$

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x^0 y^0 + x^1 y^1 + \dots + x^n y^n \quad (2.1)$$

where the non-zero vectors

$$\mathbf{x} = (x^0, x^1, \dots, x^n) \in \mathbf{V}^{n+1} \text{ and } \mathbf{y} = (y^0, y^1, \dots, y^n) \in \mathbf{V}^{n+1},$$

are determined up to real factors and they represent points in  $\mathcal{P}^n(\mathbf{R})$ .  $\mathbf{H}^n$  is represented as the interior of the absolute quadratic form

$$Q = \{[\mathbf{x}] \in \mathcal{P}^n \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0\} = \partial \mathbf{H}^n \quad (2.2)$$

in real projective space  $\mathcal{P}^n(\mathbf{V}^{n+1}, \mathbf{V}_{n+1})$ . All proper interior point  $\mathbf{x} \in \mathbf{H}^n$  are characterized by  $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ .

The points on the boundary  $\partial\mathbf{H}^n$  in  $\mathcal{P}^n$  represent the absolute points at infinity of  $\overline{\mathbf{H}}^n$ . Points  $\mathbf{y}$  with  $\langle \mathbf{y}, \mathbf{y} \rangle > 0$  lie outside of  $\overline{\mathbf{H}}^n$  and are called outer points of  $\mathbf{H}^n$ . Let  $X([\mathbf{x}]) \in \mathcal{P}^n$  a point;  $[\mathbf{y}] \in \mathcal{P}^n$  is said to be conjugate to  $[\mathbf{x}]$  relative to  $Q$  when  $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ . The set of all points conjugate to  $X([\mathbf{x}])$  form a projective polar hyperplane

$$\text{pol}(X) := \{[\mathbf{y}] \in \mathcal{P}^n \mid \langle \mathbf{x}, \mathbf{y} \rangle = 0\}. \quad (2.3)$$

Hence the bilinear form by (2.1) induces a bijection (linear polarity  $\mathbf{V}^{n+1} \rightarrow \mathbf{V}_{n+1}$ ) from the points of  $\mathcal{P}^n$  onto its hyperplanes.

Point  $X[\mathbf{x}]$  and the hyperplane  $\alpha[\mathbf{a}]$  are called incident if the value of the linear form  $\mathbf{a}$  on the vector  $\mathbf{x}$  is equal to zero; i.e.,  $\mathbf{x}\mathbf{a} = 0$  ( $\mathbf{x} \in \mathbf{V}^{n+1} \setminus \{0\}$ ,  $\mathbf{a} \in \mathbf{V}_{n+1} \setminus \{0\}$ ). Straight lines in  $\mathcal{P}^n$  are characterized by the 2-subspaces of  $\mathbf{V}^{n+1}$  or  $(n-1)$ -spaces of  $\mathbf{V}_{n+1}$  (see e.g. in [12]).

In this paper we set the sectional curvature of  $\overline{\mathbf{H}}^n$ ,  $K = -k^2$ , to be  $k = 1$ . The distance  $s$  of two proper points  $(\mathbf{x})$  and  $(\mathbf{y})$  is calculated by the formula:

$$\cosh s = \frac{-\langle \mathbf{x}, \mathbf{y} \rangle}{\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}}. \quad (2.4)$$

The foot point  $Y(\mathbf{y})$  of the perpendicular, dropped from the point  $X(\mathbf{x})$  on the plane  $(\mathbf{u})$ , has the following form:

$$\mathbf{y} = \mathbf{x} - \frac{\langle \mathbf{x}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}. \quad (2.5)$$

The length  $l(x)$  of a horocycle arc to a chord segment  $x$  is determined by the classical formula due to J. Bolyai:

$$l(x) = 2 \sinh \frac{x}{2}. \quad (2.6)$$

The volume of the horoball sectors in the  $n$ -dimensional hyperbolic space  $\overline{\mathbf{H}}^n$  can be calculated by the formula (2.7) which is the generalization of the classical formula of J. Bolyai to higher dimensions (see [?]). If the volume of the polyhedron  $A$  on the horosphere is  $\mathcal{A}$ , the volume determined by  $A$  and the aggregate of axes drawn from  $A$  is equal to

$$V = \frac{1}{n-1} \mathcal{A}. \quad (2.7)$$

### 3 Horoball packings and their simplicial densities

#### 3.1 Formula for the classical simplicial horoball density

An  $n$ -simplex in  $\overline{\mathbf{H}}^n$  is regular if its symmetry group operates transitively on the  $k$ -dimensional faces ( $0 \leq k \leq n-1$ ). In this case, it has a unique barycenter (the fixed point of the symmetry group), and all of its edge lengths and dihedral angles are of equal measure, respectively. Let  $T_{reg}^\infty(2\alpha^n) \subset \overline{\mathbf{H}}^n$  denote a regular  $n$ -simplex of dihedral angles  $2\alpha^n \in [0, \pi]$ . For hyperbolic simplicial  $n$ -volume, there are explicit formulas in terms of the dihedral angles only for  $n \leq 6$ . In case  $n = 2$  the area of an ideal triangle is  $\pi$  and for the volume of an ideal regular simplex  $Vol(T_{reg}^\infty(2\alpha_\infty^n))$  ( $n \geq 3$ ) in the  $n$ -dimensional hyperbolic space is due to J. Milnor [15].

**Theorem 3.1 (J. Milnor)** *Let  $\beta = \frac{n+1}{2}$ . Then, the ideal regular simplex volume  $Vol(T_{reg}^\infty)$  in the  $n$ -dimensional hyperbolic space ( $n \geq 3$ ), is given by*

$$Vol(T_{reg}^\infty) = \sqrt{n} \sum_{k=0}^{\infty} \frac{\beta(\beta+1) \dots (\beta+k-1)}{(n+2k)!} A_{n,k} \quad \text{where} \quad (3.1)$$

$$A_{n,k} = \sum_{i_0+\dots+i_n=k, \ i_k \geq 0} \frac{(2i_0)! \dots (2i_n)!}{i_0! \dots i_n!}.$$

The density of  $n+1$  mutually tangent horoballs  $B_i$  (the horoballs are in the same type) with respect to the regular simplex  $Vol(T_{reg}^\infty(2\alpha_\infty^n))$  formed by their base points is given by (1.3). The classical universal density upper bound that is due to K. Böröczky can be derived from this arrangement in the  $n$ -dimensional hyperbolic space  $\overline{\mathbf{H}}^n$ . In [8] R. Kellerhals have proved the following

**Theorem 3.2 (R. Kellerhals)** *The simplicial horoball density  $d_n(\infty)$  ( $n \geq 3$ ) is given by*

$$d_n(\infty) = \frac{n+1}{n-1} \frac{n}{2^{n-1}} \prod_{k=2}^{n-1} \left( \frac{k-1}{k+1} \right)^{\frac{n-k}{2}} \frac{1}{Vol(T_{reg}^\infty(2\alpha_\infty^n))} \quad (3.2)$$

**Remark 3.3** *In the hyperbolic plane  $\overline{\mathbf{H}}^2$   $d_2(\infty) = \frac{3}{\pi}$ .*

### 3.2 Formula for the generalized simplicial horoball density

The aim of this section is to determine the optimal packing arrangements  $\mathcal{B}_{opt}^n$  and their densities for the totally asymptotic simplices in  $\overline{\mathbf{H}}^n$  ( $n \geq 2$ ). We will use the consequences of the following Lemma (see [18] in 3-dimensions):

**Lemma 3.4** *Let  $B_1$  and  $B_2$  denote two horoballs with ideal centers  $C_1$  and  $C_2$  respectively in the  $n$ -dimensional hyperbolic space ( $n \geq 2$ ). Take  $\tau_1$  and  $\tau_2$  to be two congruent  $n$ -dimensional convex pyramid-like regions, with vertices  $C_1$  and  $C_2$ . Assume that these horoballs  $B_1(x)$  and  $B_2(x)$  are tangent at point  $I(x) \in C_1C_2$  and  $C_1C_2$  is a common edge of  $\tau_1$  and  $\tau_2$ . We define the point of contact  $I(0)$  such that the following equality holds for the volumes of horoball sectors:*

$$V(0) := 2\text{vol}(B_1(0) \cap \tau_1) = 2\text{vol}(B_2(0) \cap \tau_2).$$

*If  $x$  denotes the hyperbolic distance between  $I(0)$  and  $I(x)$ , then the function*

$$V(x) := \text{vol}(B_1(x) \cap \tau_1) + \text{vol}(B_2(x) \cap \tau_2) = \frac{V(0)}{2}(e^{(n-1)x} + e^{-(n-1)x})$$

*strictly increases as  $x \rightarrow \pm\infty$ .*

**Proof:** Let  $\mathcal{L}$  and  $\mathcal{L}'$  be parallel horocycles with centre  $C$  and let  $A$  and  $B$  be two points on the curve  $\mathcal{L}$  and  $A' := CA \cap \mathcal{L}'$ ,  $B' := CB \cap \mathcal{L}'$ . By the classical formula of J. Bolyai

$$\frac{\mathcal{H}(A'B')}{\mathcal{H}(AB)} = e^x,$$

where the horocyclic distance between  $A$  and  $B$  is denoted by  $\mathcal{H}(A, B)$ .

Then by the above formulas we obtain the following volume function:

$$\begin{aligned} V(x) &= \text{Vol}(B_1(x) \cap \tau_1) + \text{Vol}(B_2(x) \cap \tau_2) = \\ &= \frac{1}{2}V(0)\left(e^{(n-1)x} + \frac{1}{e^{(n-1)x}}\right) \end{aligned}$$

It is easy to see that this function strictly increases in the interval  $(0, \infty)$ .  $\square$

We consider horoball packings with centers located at ideal vertices of an totally asymptotic regular simplex  $T_{reg}^\infty = E_0E_1E_2 \dots E_n$  in the  $n$ -dimensional hyperbolic space  $\overline{\mathbf{H}}^n$  ( $n \geq 2$ ).

We consider the following two basic horoball configurations  $\mathcal{B}_n^i$ , ( $i = 1, 2$ ):

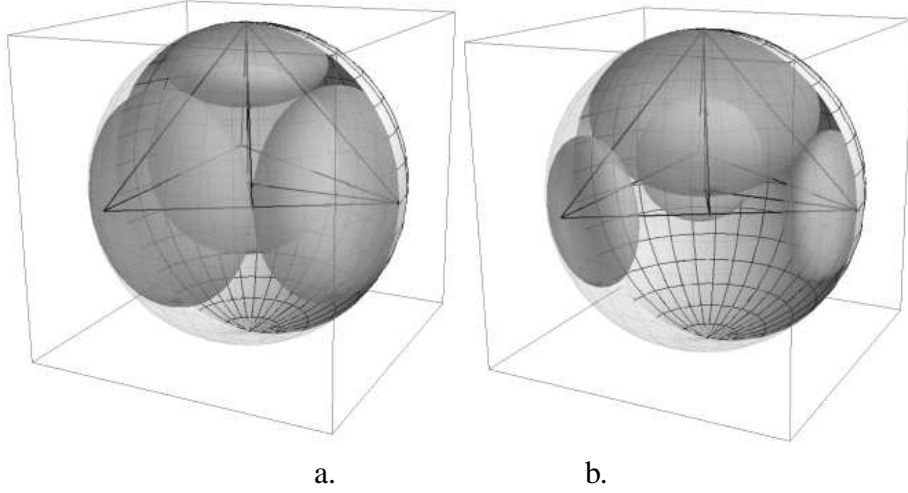


Figure 1: Two optimal horoball arrangements of  $(3, 3, 6)$  tiling.

1. All  $n + 1$  horoballs are of the same type and the adjacent horoballs touch each other at the "midpoints" of each edge. This horoball arrangement is denoted by  $\mathcal{B}_0^n$ . We define the point of tangency of two horoballs  $B_0$  and  $B_n$  on side  $E_0E_n$  to be  $I(0)$  so that the following equality holds:

$$V(0) := (n+1) \cdot \text{Vol}(B_0(0) \cap T_{reg}^\infty) = (n+1) \cdot \text{Vol}(B_n(0) \cap T_{reg}^\infty) = (n+1) \cdot V_0.$$

In the hyperbolic plane ( $n = 2$ )  $V_0 = 1$  and we obtain applying the formula

(3.2) that for dimensions  $n \geq 3$   $V_0 = \frac{1}{n-1} \frac{n}{2^{n-1}} \prod_{k=2}^{n-1} \left( \frac{k-1}{k+1} \right)^{\frac{n-k}{2}}$  in the  $n$ -dimensional hyperbolic space.

2. One horoball of the "maximally large" type centered at  $E_n$ . The large horoball  $B_n$  tangents the opposite face  $E_0E_1E_2 \dots E_{n-1}$  of  $T_{reg}^\infty$  and it determines the other  $n$  horoballs touching the "large horoball". The point of tangency of  $B_n$  and  $B_0$  along segment  $I(0)E_0$  is denoted by  $I(x_1)$  where  $x_1$  is the hyperbolic distance between  $I(0)$  and  $I(x_1)$ . This horoball arrangement is denoted by  $\mathcal{B}_1^n$ .

Due to symmetry considerations it is sufficient to consider the cases when one horoball extends from the "midpoint" of an edge until it touches the opposite side of the cell. Moreover, consider that point  $I(x)$  is on edge  $E_0E_n$ . This point is

where the horoballs  $B_i(x)$ , ( $i = 0, n$ ) are tangent at point  $I(x) \in E_0 I(0)$ . Then  $x$  is the hyperbolic distance between  $I(0)$  and  $I(x)$ . It is easy to see that we have to study that case where  $x \in [0, x_1]$  and horoball  $B_n$  touches the horoballs  $B_i$  ( $i = 1, 2, \dots, n-1$ ).

In this case the function  $V(x)$  can be computed by the following formula

$$V(x) := n \cdot \text{Vol}(B_0(x) \cap T_{reg}^\infty) + \text{Vol}(B_3(x) \cap T_{reg}^\infty) \quad x \in [0, x_1].$$

Similarly to the Lemma 3.4, we can prove the following Lemma:

**Lemma 3.5**

$$\begin{aligned} V(x) &:= n \cdot \text{Vol}(B_0(x) \cap T_{reg}^\infty) + \text{Vol}(B_3(x) \cap T_{reg}^\infty) = \\ &= V_0(e^{(n-1)x} + n \cdot e^{-(n-1)x}), \quad x \in [0, q_n], \end{aligned}$$

and the maxima of function  $V(x)$  are realized in point  $I(0)$  if  $q_n \leq \frac{1}{n-1} \log n$  or at the point  $I(q_n)$  if  $q_n \geq \frac{1}{n-1} \log n$  ( $n \geq 2$ ).

**Proof:** The second derivative of  $V(x)$  is positive for all  $n \geq 2$ , thus it is strictly convex function and so, its maximum is achieved at  $I(0)$  or at the point  $I(x_1)$ . Moreover,  $V(0) = V(\frac{1}{n-1} \log n)$  and if  $x = \frac{1}{2(n-1)} \log n$  then  $V(x)$  is minimal.  $\square$

We consider a totally asymptotic regular tetrahedron  $T_{reg}^\infty = E_0 E_1 \dots E_n$  and place the horoball centers at vertices  $E_0, \dots, E_n$ . We vary the types of the horoballs so that they satisfy our constraints of non-overlap. The packing density is obtained by Definition 1.2. The dihedral angles of the above tetrahedron at the edges are equal of measure  $2\alpha_\infty^n = \arccos\left(\frac{1}{n-1}\right)$  (see [8]).

We introduce a Euclidean projective coordinate system to the tetrahedron  $T_{reg}^\infty$  such that the center of the face  $E_0 E_1 E_2 \dots, E_{n-1}$  coincide with the center of the model  $O(1, 0, 0, \dots, 0, 0)$  moreover, we assume that  $E_0 \sim (1, 0, 0, \dots, 1, 0)$  and  $E_n \sim (1, 0, 0, \dots, 0, 1)$ .

R. Kellerhals in [9] have proved, that the in-radius  $\rho_n$  of an  $n$  dimensional regular ideal simplex can be computed by the next formula:

$$\cosh(\rho_n) = \sqrt{\frac{2n}{n+1}} \cos(\alpha_\infty^n) = \frac{n}{\sqrt{(n-1)(n+1)}}, \quad (3.3)$$

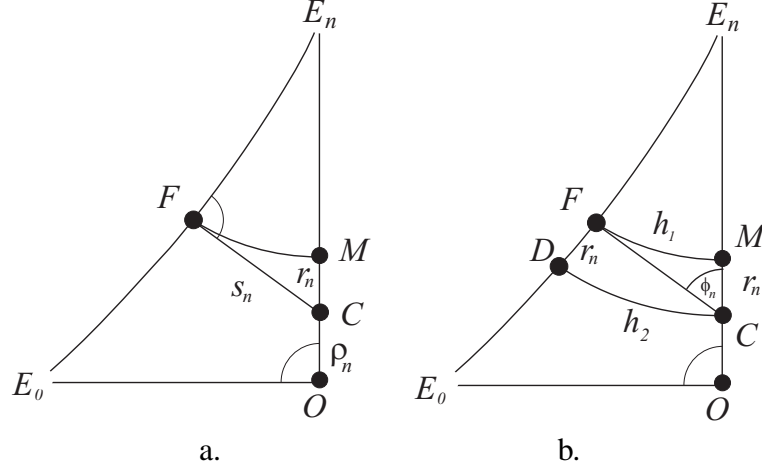


Figure 2:

therefore the coordinates of the center  $C \sim \alpha(1, 0, 0, \dots, 0, 0) + (1, 0, 0, \dots, 0, 1) = (\alpha + 1, 0, 0, \dots, 0, 1)$  of  $T_{reg}^\infty$  can be determined by formulas (2.4) and (3.3):

$$\cosh \rho_n = \cosh OC = \frac{n}{\sqrt{(n-1)(n+1)}} = \frac{1}{\sqrt{1 - \frac{1}{(\alpha+1)^2}}} \Rightarrow$$

$$C \sim (1, 0, 0, \dots, 0, \frac{1}{n}). \quad (3.4)$$

Let the foot-point of perpendicular dropped from  $C$  onto the straight line  $E_0 E_n$  be denote by  $F$  which is the common point of the horoballs  $B_n \in \mathcal{B}_0$  and  $B_0 \in \mathcal{B}_0$  centered at  $E_n$  and  $E_0$ . The hyperbolic distance  $s_n = CF$  between the point  $C(c)$  and the straight line  $E_0 E_n$ , given by  $(u)$ , can be computed by the following formula (see Fig. 2):

$$\sinh(s_n) = \frac{|\langle \mathbf{x}, \mathbf{u} \rangle|}{\sqrt{-\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{u}, \mathbf{u} \rangle}}, \quad (3.5)$$

where  $\mathbf{u}$  is the pol of the line  $E_0 E_n$  if we restrict the polarity to the plane  $E_0 E_n O$  (see Fig. 2). The parallel distance of the angle  $\phi_n = E_n C F \angle$  is  $s_n$  therefore we obtain by the classical formula of J. Bolyai and by formula (3.5) the following equation (see Fig. 2):

$$\sinh(s_n) = \cot(\phi_n) = \frac{n-1}{\sqrt{(n+1)(n-1)}}. \quad (3.6)$$

We consider a horocycle  $\mathcal{H}_2$  through the point  $C$  with center  $E_n$  in the plane  $E_n E_0 O$  and the point  $\mathcal{H}_2 \cap E_0 E_n$  is denoted by  $D$ . The horocyclic distances between points  $F, M$  and  $C, D$  are denoted by  $h_1$  and  $h_2$ . By means of formula of J. Bolyai using the formula (3.6) yields

$$\begin{aligned} \frac{h_2}{h_1} &= e^{r_n} = \frac{1}{\sin(\phi)} \Rightarrow \\ r_n &= \log \left( \frac{1}{\sin(\phi_n)} \right) = \log \left( \sqrt{\cot^2(\phi_n) + 1} \right) = \log \left( \sqrt{\frac{2n}{n+1}} \right). \end{aligned} \quad (3.7)$$

The in-radius  $\rho_n$  of an  $n$  dimensional regular ideal simplex is by the formula (3.3)

$$\rho_n = \log \left( \sqrt{\frac{n+1}{n-1}} \right), \quad (3.8)$$

therefore, the hyperbolic distance between the points  $O$  and  $M$  is

$$q_n = r_n + \rho_n = \log \left( \sqrt{\frac{2n}{n-1}} \right). \quad (3.9)$$

From the Lemma 3.5 follows, that the generalized simplicial horoball density for the regular totally asymptotic tetrahedra is maximal at the  $\mathcal{B}_0$  horoball configuration if  $q_n \leq \frac{1}{n-1} \log n$  or at the  $\mathcal{B}_1$  horoball arrangement if  $q_n \geq \frac{1}{n-1} \log n$ .

It is easy to see, that if  $n = 2, 3$  then  $q_n = \log \left( \sqrt{\frac{2n}{n-1}} \right) = \frac{1}{n-1} \log n$  and if  $n \geq 4$  then  $q_n > \frac{1}{n-1} \log n$ .

Finally, we obtain by Lemma 3.5 and formula (3.9) the main results:

**Theorem 3.6** *For  $n = 2, 3$  the maximal generalized simplicial horoball density to the regular totally asymptotic simplex in the  $n$ -dimensional hyperbolic space is realized at horoball arrangements  $\mathcal{B}_0^n$  and  $\mathcal{B}_1^n$ , as well and if  $n \geq 4$  then the maximal density is attained at the horoball configuration  $\mathcal{B}_1^n$ .*

**Theorem 3.7** *The maximal generalized simplicial horoball density for the regular totally asymptotic simplex in the  $n$ -dimensional hyperbolic space can be determined by the following formula ( $n \geq 3$ ):*

$$\begin{aligned} \delta(\mathcal{B}_{opt}^n) &= \frac{1}{n-1} \frac{n}{2^{n-1}} \prod_{k=2}^{n-1} \left( \frac{k-1}{k+1} \right)^{\frac{n-k}{2}} \cdot \\ &\cdot \left( \sqrt{\frac{2n}{n-1}}^{(n-1)} + n \cdot \sqrt{\frac{2n}{n-1}}^{(-n+1)} \right) \frac{1}{Vol(T_{reg}^\infty(2\alpha_\infty^n))} \end{aligned} \quad (3.10)$$

where  $Vol(T_{reg}^\infty(2\alpha_\infty^n))$  denotes the ideal regular  $n$ -simplex volume.

- Remark 3.8** 1. The optimal horocycle packings in the 2-dimensional hyperbolic space have the following densities:  $\delta(\mathcal{B}_0^2) = \delta(\mathcal{B}_1^2) = \frac{3}{\pi} \approx 0.95493..$
2. In the 4-dimensional hyperbolic space the classical Böröczky's upper bound related to the horoball arrangement  $\mathcal{B}_0^4$  is  $\approx 0.73046$ , but the new optimal generalized simplicial horoball density at ball configuration  $\mathcal{B}_1^4$  is  $\approx 0.77038$ .

The above results show, that the discussion of the densest horoball packings in the  $n$ -dimensional hyperbolic space with congruent horoballs in different types is not settled.

Optimal sphere packings in other homogeneous Thurston geometries represent a class of open mathematical problems. For these non-Euclidean geometries only very few results are known [21], [22]. Detailed studies are the objective of ongoing research.

## References

- [1] Bezdek, K. Sphere Packings Revisited, *European Journal of Combinatorics* (2006) **27/6** , 864–883.
- [2] Bowen, L. - Radin, C. Optimally Dense Packings of Hyperbolic Space, *Geometriae Dedicata*, (2004) **104** , 37–59.
- [3] Böhm, J. - Hertel, E. *Polyedergeometrie in  $n$ -dimensionalen Räumen konstanter Krümmung*, Birkhäuser, Basel (1981).
- [4] Böröczky, K. Packing of spheres in spaces of constant curvature, *Acta Math. Acad. Sci. Hungar.*, (1978) **32** , 243–261.
- [5] Böröczky, K. - Florian, A. Über die dichteste Kugelpackung im hyperbolischen Raum, *Acta Math. Acad. Sci. Hungar.*, (1964) **15** , 237–245.
- [6] Coxeter, H. S. M. Regular honeycombs in hyperbolic space, *Proceedings of the international Congress of Mathematicians, Amsterdam*, (1954) **III** , 155–169.

- [7] Fejes Tóth, G. - Kuperberg, G. - Kuperberg, W. Highly Saturated Packings and Reduced Coverings, *Monatshefte für Mathematik*, (1998) **125/2**, 127–145.
- [8] Kellerhals, R. Ball packings in spaces of constant curvature and the simplicial density function, *Journal für reine und angewandte Mathematik*, (1998) **494**, 189–203.
- [9] Kellerhals, R. Regular simplices and lower volume bounds for hyperbolic  $n$ -manifolds, *Ann. Global Anal. Geom.*, (1995) **13**, 377–392.
- [10] Kozma, T. R. - Szirmai, J. Optimally dense packings for fully asymptotic Coxeter tilings by horoballs of different types, *Submitted to Monatshefte für Mathematik*, 2011.
- [11] Molnár, E. Klassifikation der hyperbolischen Dodekaederpflasterungen von flächentransitiven Bewegungsgruppen, *Mathematica Pannonica*, (1993) **4/1**, 113–136.
- [12] Molnár, E. The projective interpretation of the eight 3-dimensional homogeneous geometries, *Beiträge Alg. Geom.*, (*Contr. Alg. Geom.*), (1997) **38/2**, 261–288.
- [13] Molnár, E. - Prok, I. - Szirmai, J. Classification of tile-transitive 3-simplex tilings and their realizations in homogeneous spaces. A. Prékopa and E. Molnár, (eds.). *Non-Euclidean Geometries, János Bolyai Memorial Volume, Mathematics and Its Applications*, Springer (2006) Vol. **581**, 321–363.
- [14] Marshall, T. H. Asymptotic Volume Formulae and Hyperbolic Ball Packing, *Annales Academiæ Scientiarum Fennicæ: Mathematica*, (1999) **24**, 31–43.
- [15] Milnor, J. Geometry, Collected papers, *Publish or Perish*, (1994) Vol **1**.
- [16] Radin, C. The symmetry of optimally dense packings, *Non-Euclidean Geometries*, eds. A. Prekopa, E. Molnar, pp. 197–207, *Springer Verlag*, (2006).

- [17] Rogers, C. A. Packing and covering, *Cambridge University Press*, (1964).
- [18] Szirmai, J. Horoball packings for the Lambert-cube tilings in the hyperbolic 3-space, *Beiträge Alg. Geom.*, (*Contr. Alg. Geom.*), (2005) **46**(1) , 43-60.
- [19] Szirmai, J. The optimal ball and horoball packings of the Coxeter tilings in the hyperbolic 3-space *Beiträge Alg. Geom.*, (*Contr. Alg. Geom.*), (2005) **46/2**, 545–558.
- [20] Szirmai, J. The optimal ball and horoball packings to the Coxeter honeycombs in the hyperbolic d-space *Beiträge Alg. Geom.*, (*Contr. Alg. Geom.*), (2007) **48/1**, 35–47.
- [21] Szirmai, J. The densest geodesic ball packing by a type of Nil lattices *Beiträge Alg. Geom.*, (*Contr. Alg. Geom.*), (2007) **48/2**, 383–397.
- [22] Szirmai, J. The densest translation ball packing by fundamental lattices in Sol space *Beiträge Alg. Geom.*, (*Contr. Alg. Geom.*), (2010) **51/2**, 353–373.
- [23] Szirmai, J. Horoball packings and their densities by generalized simplicial density function in the hyperbolic space *Acta Mathematica Hungarica*, (2011) to appear.
- [24] Zagier, D. The Dilogarithm Function *Frontiers in Number Theory, Physics, and Geometry II : On Conformal Field Theories, Discrete Groups and Renormalization*, Springer, (2007) 3–65.